Math 4210 Assignment 2

Due date: November 15th, 2022, 11:59pm

Problem 1 [50 marks]

Let us consider a continuous time market, where the continuously compounded interest rate is $r > 0$, the risky asset $S = (S_t)_{0 \le t \le T}$ follows the Black-Scholes model with drift μ and volatility σ, so that S follows the dynamic

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

with Brownian motion B.

a) (5 points) Give the expression of S_t as function of (t, B_t).

b) (5 points) Let $\mathbb{Q}$ denote the (equivalent) risk neutral probability, and $B^{\mathbb{Q}}$ denote the corresponding Brownian motion under $\mathbb{Q}$. Write down the dynamic of S under $\mathbb{Q}$, and then give the expression of S_t as function of $(t, B_t^{\mathbb{Q}})$.

c) (20 points) We consider a derivative option with payoff $1_{\{S_T > K\}}$. Compute the following expectation

$$\mathbb{E}^{\mathbb{Q}}\big[e^{-rT} 1_{\{S_T > K\}}\big].$$

Warning: Please note the sign difference with the midterm question.

d) (20 points) Similarly, we consider a derivative option with payoff $|S_T - K|$ (absolute value of the difference), for some constant $K > 0$. Compute the following expectation

$$\mathbb{E}^{\mathbb{Q}}\big[e^{-rT} \big|S_T - K\big|\big].$$

Note: Please express the above expectation values in terms of the parameters r, μ, σ, T and K. You may use $\Phi : \mathbb{R} \longrightarrow [0, 1]$ to denote the cumulative distribution function of the standard Gaussian distribution $N(0, 1)$.

Problem 2 [50 marks]

We consider a continuous time market, where the interest rate $r = 0$, and the risky asset $S = (S_t)_{0 \le t \le T}$ follows the Black-Scholes model with initial value $S_0 = 1$, drift μ and volatility $\sigma > 0$ (without any dividend), so that

$$dS_t = \mu S_t dt + \sigma S_t dB_t.$$

a) (5 points) A self-financing portfolio is given by (x, ϕ), where x represents the initial wealth of the portfolio, and ϕ_t represents the number of risky asset in the portfolio at time t. Let $(\Pi_t^{x,\phi})_{t \in [0,T]}$ be the wealth process of the portfolio, write down the dynamic of $\Pi^{x,\phi}$ in form of

$$d\Pi_t^{x,\phi} \;=\; \alpha_t dt + \beta_t dB_t, \text{ for some (to be founded) process } (\alpha, \beta).$$

b) (5 points) There exists a unique risky-neutral probability $\mathbb{Q}$, together with a Brownian motion $B^{\mathbb{Q}}$ under the probability measure $\mathbb{Q}$.

Please give the expression of the process S_t as a function of $(t, B_t^{\mathbb{Q}})$.

c) (40 points) We consider a derivaitive option with payoff $g(S_T) = \log(S_T)$ at maturity T.

- (5 points) Compute the value

$$V_0 \;:=\; \mathbb{E}^{\mathbb{Q}}\big[\log(S_T)\big].$$

- (25 points) Let $v(t,x) := \log(x) - \frac{1}{2}\sigma^2(T-t)$, compute $\partial_t v$, $\partial_x v$ and $\partial^2_{xx} v$, and check that v satisfies the equation

$$\partial_t v(t,x) + \frac{1}{2}\sigma^2 x^2 \partial^2_{xx} v(t,x) \;=\; 0, \quad (t,x) \in [0,T) \times (0,\infty),$$

with terminal condition $v(T,x) = \log(x)$ for all $x \in (0,\infty)$.

- (10 points) Remember that S_t is a function of (t, B_t), apply the Itô formula on $v(t, S_t)$ to deduce that

$$\log(S_T) = V_0 + \int_0^T \phi_t dS_t, \text{ where } \phi_t := \frac{1}{S_t}.$$

Finally, deduce the (no-arbitrage) price of the derivative option with payoff $g(S_T) = \log(S_T)$.